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METRIC CLASSIFICATION OF CONICS AND QUADRICS BY MEANS OF RANK

BY GEORGE RUTLEDGE

IT is well known that, under non-singular linear substitutions, the rank of the determinant of a homogeneous quadratic form, in n variables, is an absolute invariant. It follows directly from this that the rank of the determinant of a non-homogeneous quadratic form, in $n-1$ variables, and the rank of the determinant of its homogeneous quadratic part, are absolute invariants under substitutions of the type

$$x_i = l_{i,1}\, y_1, + l_{i,2} y_2 + \cdots + l_{i,\, n-1}\, y_{n-1} + l_{i,\, n} \qquad (i = 1, 2, \cdots, n-1),$$

where the determinant of the homogeneous part of the substitution is non-singular.

Hence, in particular, the rank of the determinant of the equation of a conic or quadric, and the rank of the determinant of its homogeneous quadratic part are absolute invariants under all rigid displacements of the conic or quadric. This fact may be made use of to determine the type of a conic or quadric by a mere inspection of its equation, without any transformation.

It is shown in the text-books,* that the equation of a quadric may always be reduced, by rigid displacements, to the form,

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + 2kx_3 + c = 0,$$

where either k or λ_3 is zero, and either k or c is zero.

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* For instance, Fort-Schlömilch, *Lehrbuch der analytischen Geometrie*, vol. 2, p. 207.

For a purely analytic treatment of this reduction, cf. Bromwich. *Quadratic Forms and their Classification by Means of Invariant Factors* (Cambridge, England, 1906), p. 72.

There are, then, three types of the reduced equation, as follows:

First type: Reduced equation, $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + c = 0, \quad c \neq 0$

I_1	$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$	Central Quadric
I_2	$\lambda_1 \cdot \lambda_2 \neq 0,\ \lambda_3 = 0$	Elliptic or Hyperbolic Cylinder
I_3	$\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 = 0$	Two Parallel Planes

Second type: Reduced equation, $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$

II_1	$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$	Cone
II_2	$\lambda_1 \cdot \lambda_2 \neq 0,\ \lambda_3 = 0$	Two Intersecting Planes
II_3	$\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 = 0$	Two Coincident Planes

Third type: Reduced equation, $\lambda_1 x_1^2 + \lambda_2 x_2^2 + 2kx_3 = 0, \ k \neq 0$

III_2	$\lambda_1 \cdot \lambda_2 \neq 0,\ \lambda_3 = 0$	Paraboloid
III_3	$\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 = 0$	Parabolic Cylinder

If, in each case, we determine the rank of the determinant

$$\begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & k \\ 0 & 0 & k & c \end{vmatrix}$$

this rank will be the same as the rank of the corresponding determinant of the original unreduced equation, and similarly for the rank of the principal first minor in the upper left-hand corner.

The minors which do not vanish identically are:

$$\begin{vmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & k \\ 0 & k & c \end{vmatrix} \quad \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & k \\ 0 & k & c \end{vmatrix} \quad \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & c \end{vmatrix} \quad \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & k \end{vmatrix} \quad \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix}$$

$$\begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} \quad \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix} \quad \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \quad \begin{vmatrix} \lambda_1 & 0 \\ 0 & c \end{vmatrix} \quad \begin{vmatrix} \lambda_1 & 0 \\ 0 & k \end{vmatrix} \quad \begin{vmatrix} \lambda_2 & 0 \\ 0 & c \end{vmatrix} \quad \begin{vmatrix} \lambda_2 & 0 \\ 0 & k \end{vmatrix} \quad \begin{vmatrix} \lambda_3 & k \\ k & c \end{vmatrix}$$

An examination of the above determinant and minors yields the following result, where R = the rank of the determinant of the *original* unreduced equation, and R_1 = the rank of its principal first minor in the upper left-hand corner:

R	R_1	TYPE	
4	3	I_1	Central Quadric
4	2	III_2	Paraboloid
3	3	II_1	Cone
3	2	I_2	Elliptic or Hyperbolic Cyl.
3	1	III_3	Parabolic Cylinder
2	2	II_2	Two Intersecting Planes
2	1	I_3	Two Parallel Planes
1	1	II_3	Two Coincident Planes
1	0		Equation linear

The necessary changes in the case of conics will be recognized without difficulty. We obtain:

3	2	I_1	Ellipse or Hyperbola
3	1	III_2	Parabola
2	2	II_1	Two Intersecting Lines
2	1	I_2	Two Parallel Lines
1	1	II_2	Two Coincident Lines
1	0		Equation linear

As an example of the foregoing method, consider the quadric*

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* Discussed by the ordinary method in C. Smith, *Solid Geometry*, p. 63.

$$4x^2 + y^2 + 4z^2 - 4yz + 8zx - 4xy + 2x - 4y + 5z + 1 = 0.$$

The determinant to be considered is

$$\begin{vmatrix} 4 & -2 & 4 & 1 \\ -2 & 1 & -2 & -2 \\ 4 & -2 & 4 & \frac{5}{2} \\ 1 & -2 & \frac{5}{2} & 1 \end{vmatrix}$$

Adding twice the second column to the first and to the third, which does not change the rank of either of the determinants with which we are concerned,[†] we have,

$$\begin{vmatrix} 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & 0 & \frac{5}{2} \\ -3 & -2 & -\frac{3}{2} & 1 \end{vmatrix}$$

whence it is evident that the determinant of the original equation is of rank 3, and that its principal first minor in the upper left-hand corner is of rank 1.

The quadric represented is consequently a parabolic cylinder.

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[†] Bôcher, *Introduction to Higher Algebra*, p. 55.